INTEGRATION OVER THE PAULI QUANTUM GROUP

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ABSTRACT. We prove that the Pauli representation of the quantum permutation algebra $A_s(4)$ is faithful. This provides the second known model for a free quantum algebra. We use this model for performing some computations, with the main result that at the level of laws of diagonal coordinates, the Lebesgue measure appears between the Dirac mass and the free Poisson law.

INTRODUCTION

The notion of free quantum group appeared in Wang's papers [17], [18]. The idea is that given a compact group $G \subset U_n$, the matrix coordinates $u_{ij} \in C(G)$ commute with each other, and satisfy certain relations R. One can define then the universal algebra A generated by abstract variables u_{ij}, subject to the relations R. In certain situations we get a Hopf algebra in the sense of Woronowicz [20], and we have the heuristic formula $A = C(G^+)$. Here G^+ is a compact quantum group, called free version of G. The free version is not unique, because it depends on R.

This construction is not axiomatized, and there are only a few examples:

(1) The first two groups, considered in [17], are $G = U_n$ and $G = O_n$. The symmetric group $G = S_n$ was considered a few years later, in [18]. The corresponding quantum groups have been intensively studied since then. However, the "liberation" aspect of the construction $G \to G^+$ was understood only recently: the idea is that with $n \to \infty$ the integral geometry of G is governed by probabilistic independence, while that of G^+, by freeness in the sense of Voiculescu. This follows from [7], [8], [12].

(2) A number of subgroups $G \subset S_n$ have been considered recently, most of them being symmetry groups of graphs, or of other combinatorial structures. Some freeness appears here as well, for instance in certain situations the operation $G \to G^+$ transforms usual wreath products into free wreath products. However, the main result so far goes somehow in the opposite sense: for certain classes of groups, with $n \to \infty$ we have $G^+ = G$. See [4], [5], [6].

Summarizing, the study of free quantum groups focuses on several ad-hoc constructions $G \to G^+$. The following two questions are of particular interest in connection with the axiomatization problem: when does G^+ collapse to G? when is G^+

a "true" free version of G? There are already several answers to these questions, most of them being of asymptotic nature. Work here is in progress.

In this paper we study the free quantum group associated to $G = S_4$. This corersponds to the Hopf algebra $A_s(4)$ generated by 16 abstract variables, subject to certain relations, similar to those satisfied by the coordinates of $S_4 \subset U_4$. These relations, discovered in [18], are known under the name "magic condition".

We should mention first that for $G = S_n$ with $n = 1, 2, 3$ we have $A_s(n) = C(S_n)$, and there is no quantum group to be studied. The other fact is that for $n \geq 5$ the algebra $A_s(n)$ is too big, for instance it is not amenable, so a detailed study here would be much more difficult, and would require new ideas.

The interest in $A_s(4)$ comes from the fact that the space formed by 4 points is the simplest one to have a non-trivial quantum automorphism group. This suggests that $A_s(4)$ itself might be the "simplest" Hopf algebra, and we believe indeed that it is so. This algebra corresponds for instance to the situation "index 4, graph A_∞", known from the work of Jones to be the basic one. See [3].

We have three results about $A_s(4)$:

(1) An explicit realization of the magic condition is found in [9], by using the Pauli matrices. This gives a representation $\pi : A_s(4) \to C(SU_2, M_4(\mathbb{C}))$, shown there to be faithful in some weak sense. At that time there was no technique for proving or disproving the fact that π is faithful. This problem can be solved now by using integration techniques, inspired from [10], [11], [12], [19] and introduced in [7], [8], and our result is that π is faithful. In other words, we have a model for the abstract algebra $A_s(4)$. This is the second known model for a free quantum algebra, the first one being a certain embedding $A_u(2) \to C(\mathbb{T}) * C(SU_2)$, found in [1].

(2) The model can be used for working out in detail the integral geometry of the corresponding quantum group. As in our previous work [7], [8], the problem that we consider is that of computing laws of sums of diagonal coordinates u_{ii}. Each such coordinate is a projection of trace $1/4$, and it is known from [2] that the sum $u_{11} + u_{22} + u_{33} + u_{44}$ is free Poisson on $[0, 4]$. Our main result here is that the law of $u_{11} + u_{22}$ is an average between a Dirac mass at 0, and the Lebesgue measure on $[0, 2]$. In other words, modulo 0 and rescalings, the averages $M_s = (u_{11} + \ldots + u_{ss})/s$ with $s = 1, 2, 4$ correspond to a Dirac mass, a Lebesgue measure, and a free Poisson law.

(3) We compute as well the laws of the variables interpolating between M_1, M_2 and M_2, M_4. The results here are quite technical, and the whole study belongs to a "higher order" problematics for free quantum groups. The idea is that one possible escape from asymptotic philosophy for $G = O_n$, probably adaptable to $G = U_n, S_n$ as well, would be the use of meander determinants of Di Francesco [13]. The other possible solution is via matrix models, and we use here the Pauli model in order for making some advances.

The paper is organized as follows: 1 is an introduction to $A_s(4)$, in 2-4 we discuss the Pauli representation, and in 5-9 we compute probability measures.

1. Quantum permutation groups

Let A be a C^*-algebra. That is, we have a complex algebra with a norm and an involution, such that the Cauchy sequences converge, and $||aa^*|| = ||a||^2$.

The basic example is $B(H)$, the algebra of bounded operators on a Hilbert space H. In fact, any C^*-algebra appears as subalgebra of some $B(H)$.

The key example is $C(X)$, the algebra of continuous functions on a compact space X. By a theorem of Gelfand, any commutative C^*-algebra is of this form.

Definition 1.1. *Let A be a C^*-algebra.*

  (1) *A projection of A is an element $p \in A$ satisfying $p^2 = p = p^*$.*
  (2) *Two projections $p, q \in A$ are called orthogonal when $pq = 0$.*
  (3) *A partition of the unity of A is a finite set of projections of A, which are pairwise orthogonal, and which sum up to 1.*

A projection in $B(H)$ is an orthogonal projection $P(K)$, where $K \subset H$ is a closed subspace. Orthogonality of projections corresponds to orthogonality of subspaces, and partitions of unity correspond to decompositions of H.

A projection in $C(X)$ is a characteristic function $\chi(Y)$, where $Y \subset X$ is an open and closed subset. Orthogonality of projections corresponds to disjointness of subsets, and partitions of unity correspond to partitions of X.

Definition 1.2. *A magic unitary is a square matrix $u \in M_n(A)$, whose rows and columns are all partitions of unity in A.*

A magic unitary over $B(H)$ is of the form $P(K_{ij})$, with K magic decomposition of H, in the sense that all rows and columns of K are decompositions of H. The basic examples here are of the form $K_{ij} = \mathbb{C}\,\xi_{ij}$, where ξ is a magic basis of H, in the sense that all rows and columns of ξ are bases of H.

A magic unitary over $C(X)$ is of the form $\chi(Y_{ij})$, with Y magic partition of X, in the sense that all rows and columns of Y are partitions of X. The key example here comes from a finite group G acting on a finite set X: the characteristic functions $\chi_{ij} = \{\sigma \in G \mid \sigma(j) = i\}$ form a magic unitary over $C(G)$.

In the particular case of the symmetric group S_n acting on $\{1, \dots, n\}$, we have the following presentation result, which follows from the Gelfand theorem:

Theorem 1.1. *$C(S_n)$ is the commutative C^*-algebra generated by n^2 elements χ_{ij}, with relations making (χ_{ij}) a magic unitary matrix. Moreover, the maps*

$$\begin{aligned}
\Delta(\chi_{ij}) &= \sum \chi_{ik} \otimes \chi_{kj} \\
\varepsilon(\chi_{ij}) &= \delta_{ij} \\
S(\chi_{ij}) &= \chi_{ji}
\end{aligned}$$

are the comultiplication, counit and antipode of $C(S_n)$.

In other words, when regarding S_n as an algebraic group, the relations satisfied by the n^2 coordinates are those expressing magic unitarity.

We are interested in the algebra of "free coordinates" on S_n. This is obtained by removing commutativity in the above presentation result:

Definition 1.3. $A_s(n)$ *is the C^*-algebra generated by n^2 elements u_{ij}, with relations making (u_{ij}) a magic unitary matrix. The maps*

$$\begin{aligned}
\Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj} \\
\varepsilon(u_{ij}) &= \delta_{ij} \\
S(u_{ij}) &= u_{ji}
\end{aligned}$$

are called the comultiplication, counit and antipode of $A_s(n)$.

This algebra, discovered by Wang in [18], fits into Woronowicz's quantum group formalism in [20]. In fact, the quantum group S_n^+ defined by the heuristic formula $A_s(n) = C(S_n^+)$ is a free analogue of the symmetric group S_n. This quantum group doesn't exist of course: the idea is just that various properties of $A_s(n)$ can be expressed in terms of it. As an example, the canonical quotient map $A_s(n) \to C(S_n)$ should be thought of as coming from an embedding $S_n \subset S_n^+$.

Proposition 1.1. *For $n = 1, 2, 3$ we have $A_s(n) = C(S_n)$.*

This follows from the fact that for $n \leq 3$, the entries of a $n \times n$ magic unitary matrix have to commute. Indeed, at $n = 2$ the matrix must be of the form

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

with p projection, and entries of this matrix commute. For $n = 3$ see [18].

This result is no longer true for $n = 4$, where more complicated examples of magic unitary matrices are available, for instance via diagonal concatenation:

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This example shows that the algebra generated by p, q is a quotient of $A_s(4)$. With $pq \neq qp$ we get that $A_s(4)$ is non-commutative, hence bigger that $C(S_4)$.

We have the following result, proved in [2], [3], [18].

Theorem 1.2. *The algebra $A_s(4)$ has the following properties:*

(1) *It is non-commutative, and infinite dimensional.*
(2) *It is amenable in the discrete quantum group sense.*
(3) *The fusion rules are the same as those for SO_3.*

We would like to end with a comment about the algebra $A_s(n)$ with $n \geq 5$, which won't appear in what follows. As in the case $n = 4$, this algebra is non-commutative, infinite dimensional, and has same fusion rules as SO_3.

The subtlety comes from the fact that the irreducible corepresentations have dimensions bigger than those of SO_3, and this makes this algebra non-amenable. In fact, much worse is expected to be true: there is evidence from [14] that the reduced version should be simple, and that the corresponding von Neumann algebra should be a prime II_1-factor. In other words, this algebra has bad analytic properties.

2. The Pauli representation

The purpose of this paper is to provide a detailed analytic description of $A_s(4)$. We use an explicit matrix model, coming from the Pauli matrices:

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad c_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad c_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad c_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

These are elements of SU_2. In fact, SU_2 consists of linear combinations of Pauli matrices, with points on the real sphere S^3 as coefficients:

$$SU(2) = \left\{ \sum x_i c_i \mid \sum |x_i|^2 = 1 \right\}$$

The Pauli matrices multiply according to the formulae for quaternions:

$$c_2^2 = c_3^2 = c_4^2 = -1$$
$$c_2 c_3 = -c_3 c_2 = c_4$$
$$c_3 c_4 = -c_4 c_3 = c_2$$
$$c_4 c_2 = -c_2 c_4 = c_3$$

The starting remark is that the Pauli matrices form an orthonormal basis of $M_2(\mathbb{C})$, with respect to the scalar product $< a, b >= tr(b^*a)$. Moreover, the same is true if we multiply them to the left or to the right by an element of SU_2.

We can formulate this fact in the following way.

Theorem 2.1. *For any $x \in SU_2$ the elements $\xi_{ij} = c_i x c_j$ form a magic basis of $M_2(\mathbb{C})$, with respect to the sclar product $< a, b >= \mathrm{tr}(b^*a)$.*

We fix a Hilbert space isomorphism $M_2(\mathbb{C}) \simeq \mathbb{C}^4$, and we use the corresponding identification of operator algebras $B(M_2(\mathbb{C})) \simeq M_4(\mathbb{C})$.

Associated to each element $x \in SU_2$ is the representation of $A_s(4)$ mapping u_{ij} to the rank one projection on $c_i x c_j$:

$$\pi_x : A_s(4) \to M_4(\mathbb{C})$$

This representation depends on x. For getting a faithful representation, the idea is to regard all these representations as fibers of a single representation.

Definition 2.1. *The Pauli representation of $A_s(4)$ is the map*

$$\pi : A_s(4) \to C(SU_2, M_4(\mathbb{C}))$$

mapping u_{ij} to the function $x \to$ rank one projection on $c_i x c_j$.

As a first remark, in this statement SU_2 can be replaced by $PU_2 = SO_3$. For reasons that will become clear later on, we prefer to use SU_2.

This representation is introduced in [9], with the main result that it is faithful in some weak sense. In what follows we prove that π is faithful, in the usual sense.

Consider the natural linear form on the algebra on the right:

$$\int \varphi = \int_{SU_2} \mathrm{tr} \left(\varphi(x) \right) \, dx$$

We use the following analytic formulation of faithfulness.

Proposition 2.1. *The representation π is faithful provided that*

$$\int u_{i_1 j_1} \ldots u_{i_k j_k} = \int \pi_{i_1 j_1} \ldots \pi_{i_k j_k}$$

for any choice of k and of various i, j indices, where $\pi_{ij} = \pi(u_{ij})$.

Proof. The condition in the statement is that

$$\int a = \int \pi(a)$$

for any product a of generators u_{ij}. By linearity and density this formula holds on the whole algebra $A_s(4)$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
A_s(4) & \xrightarrow{\int} & \mathbb{C} \\[2em]
\pi \downarrow & & \uparrow \mathrm{tr} \\[2em]
C\left(SU_2, M_4(\mathbb{C})\right) & \xrightarrow{\int} & M_4(\mathbb{C})
\end{array}
$$

On the other hand, we know that $A_s(4)$ is amenable in the discrete quantum group sense. This means that its Haar functional is faithful:

$$a \neq 0 \Longrightarrow \int aa^* > 0$$

Assume now that we have $\pi(a) = 0$. This implies $\pi(aa^*) = 0$, and commutativity of the above diagram gives $\int aa^* = 0$. Thus $a = 0$, and we are done. $\qquad\square$

We compute now the integral on the right in the above statement.

Consider the canonical action of SU_2 on the algebra $M_2(\mathbb{C})^{\otimes k}$, obtained as k-th tensor power of the adjoint action on $M_2(\mathbb{C})$:

$$\alpha_x(a_1 \otimes \ldots \otimes a_k) = xa_1x^* \otimes \ldots \otimes xa_kx^*$$

The following map will play an important role throughout this paper.

Definition 2.2. *We define a linear map $R : M_2(\mathbb{C})^{\otimes k} \to M_2(\mathbb{C})^{\otimes k}$ by*

$$R(c_{i_1} \otimes \ldots \otimes c_{i_k}) = \frac{1}{2}(c_{i_1} c_{i_2}^* \otimes c_{i_2} c_{i_3}^* \otimes \ldots \otimes c_{i_k} c_{i_1}^*)$$

with the convention that for $k = 1$ we have $R(c_i) = c_i c_i^*/2 = 1/2$.

To any multi-index $i = (i_1, \ldots, i_k)$ we associate the following element:

$$c_i = c_{i_1} \otimes \ldots \otimes c_{i_k}$$

With these notations, we have the following result.

Proposition 2.2. *We have the formula*

$$\int \pi_{i_1 j_1} \ldots \pi_{i_k j_k} = < R^* E R(c_j), c_i >$$

where E is the expectation under the canonical action of SU_2.

Proof. We have the following computation, where $P(\xi)$ denotes the rank one projection onto a vector ξ:

$$
\begin{aligned}
\int \pi_{i_1 j_1} \dots \pi_{i_k j_k} &= \int \pi(u_{i_1 j_1}) \dots \pi(u_{i_k j_k}) \\
&= \int P(c_{i_1} x c_{j_1}) \dots P(c_{i_k} x c_{j_k}) \\
&= \int_{SU_2} \text{tr}\left(P(c_{i_1} x c_{j_1}) \dots P(c_{i_k} x c_{j_k})\right) dx
\end{aligned}
$$

We use now the following elementary formula, valid for any sequence of norm one vectors $\xi_1, \dots, \xi_k$ in a Hilbert space:

$$
\text{Tr}\left(P(\xi_1) \dots P(\xi_k)\right) = \ <\xi_1, \xi_2><\xi_2, \xi_3> \dots <\xi_k, \xi_1>
$$

In our situation these vectors are in fact matrices, and their scalar products are given by $<\xi, \eta> = \text{tr}(\xi\eta^*)$. This gives the following formula:

$$
\begin{aligned}
\int \pi_{i_1 j_1} \dots \pi_{i_k j_k} &= \frac{1}{4} \int_{SU_2} <c_{i_1} x c_{j_1}, c_{i_2} x c_{j_2}> \dots <c_{i_k} x c_{j_k}, c_{i_1} x c_{j_1}> dx \\
&= \frac{1}{4} \int_{SU_2} \text{tr}(c_{i_1} x c_{j_1} c_{j_2}^* x^* c_{i_2}^*) \dots \text{tr}(c_{i_k} x c_{j_k} c_{j_1}^* x^* c_{i_1}^*) dx \\
&= \frac{1}{4} \int_{SU_2} \text{tr}(c_{i_2}^* c_{i_1} x c_{j_1} c_{j_2}^* x^*) \dots \text{tr}(c_{i_1}^* c_{i_k} x c_{j_k} c_{j_1}^* x^*) dx
\end{aligned}
$$

We use now the formula $c_s^* = \pm c_s$, valid for all matrices c_s. The minus signs can be rearranged, and the computation can be continued as follows:

$$
\begin{aligned}
\int \pi_{i_1 j_1} \dots \pi_{i_k j_k} &= \frac{1}{4} \int_{SU_2} \text{tr}(c_{i_2} c_{i_1}^* x c_{j_1} c_{j_2}^* x^*) \dots \text{tr}(c_{i_1} c_{i_k}^* x c_{j_k} c_{j_1}^* x^*) dx \\
&= \frac{1}{4} \int_{SU_2} \text{tr}\left(c_{i_2} c_{i_1}^* x c_{j_1} c_{j_2}^* x^* \otimes \dots \otimes c_{i_1} c_{i_k}^* x c_{j_k} c_{j_1}^* x^*\right) dx \\
&= \int_{SU_2} \text{tr}\left(R(c_i)^* \alpha_x(R(c_j))\right) dx
\end{aligned}
$$

We can interchange the trace and integral signs:

$$
\begin{aligned}
\int \pi_{i_1 j_1} \dots \pi_{i_k j_k} &= \text{tr}\left(\int_{SU_2} R(c_i)^* \alpha_x(R(c_j)) dx\right) \\
&= \text{tr}\left(R(c_i)^* \int_{SU_2} \alpha_x(R(c_j)) dx\right)
\end{aligned}
$$

Now acting by group elements, then integrating, is the same as projecting onto fixed points, so the computation can be continued as follows:

$$
\begin{aligned}
\int \pi_{i_1 j_1} \dots \pi_{i_k j_k} &= \text{tr}\left(R(c_i)^* E R(c_j)\right) \\
&= \ <ER(c_j), R(c_i)> \\
&= \ <R^* E R(c_j), c_i>
\end{aligned}
$$

This finishes the proof. $\qquad\square$

3. Some technical results

We know from Proposition 2.1 that the faithfulness of the Pauli representation is equivalent to a certain equality of integrals. Moreover, one of these integrals can be computed by using the Weingarten formula in [8]. As for the other integral, Proposition 2.2 shows that this can be computed as well, provided we have enough information about the operator R from Definition 2.2.

In this section we work out a number of technical properties of this operator R. For this purpose, we need first understand some aspects of the structure of the algebra of fixed points under the diagonal adjoint action of SU_2.

Lemma 3.1. $f = \sum_{i=1}^{4} c_i \otimes c_i^*$ is invariant under the action of SU_2.

Proof. We have $c_i = -c_i^*$ for $i = 2, 3, 4$, so the element in the statement is:

$$f = 2(c_1 \otimes c_1) - \sum_{i=1}^{4} c_i \otimes c_i$$

Since $c_1 \otimes c_1$ is invariant under SU_2, what is left to prove is that the sum on the right is invariant as well. This sum, viewed as a matrix, is:

$$\sum_{i=1}^{4} c_i \otimes c_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

But this matrix is 4 times the projection onto the determinant subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$, which is invariant under the action of SU_2. This concludes the proof. $\square$

Let $NC(k)$ be the set of noncrossing partitions of $\{1, \ldots, k\}$. Given a partition $p \in NC(k)$ and a multi-index $j = (j_1, \ldots, j_k)$, we can plug j into p in the obvious way, and we define a number $\delta_{pj} \in \{0, 1\}$ in the following way: $\delta_{pj} = 1$ if any block of p contains identical indices of j, and $\delta_{pj} = 0$ if not. See [8].

To any $p \in NC(k)$ we associate an element of $M_2(\mathbb{C})^k$, in the following way:

$$c_p = \sum_j \delta_{pj} c_j$$

Our next goal is to prove that $R(c_p)$ is invariant under the action of SU_2. We discuss first the case of the trivial partition, $0_k = \{\{1\}, \ldots, \{k\}\}$.

Lemma 3.2. $R(c_{0_k})$ is invariant under the action of SU_2.

Proof. The partition $p = 0_k$ has the particular property that $\delta_{pj} = 1$ for any multi-index $j = (j_1, \ldots, j_k)$. This gives the following formula:

$$R(c_{0_k}) = R\left(\sum_j c_j\right)$$

$$= R\left(\sum_{j_1\ldots j_k} c_{j_1} \otimes \ldots \otimes c_{j_k}\right)$$

$$= \frac{1}{2}\sum_{j_1\ldots j_k} c_{j_1}c_{j_2}^* \otimes c_{j_2}c_{j_3}^* \otimes \ldots \otimes c_{j_k}c_{j_1}^*$$

To any multi-index $j = (j_1,\ldots,j_k)$ we associate a multi-index $i = (i_1,\ldots,i_{k-1})$ in the following way: i_1 is such that $c_{i_1} = \pm c_{j_1}c_{j_2}^*$, i_2 is such that $c_{i_2} = \pm c_{j_1}c_{j_3}^*$, and so on up to the last index i_{k-1}, which is such that $c_{i_{k-1}} = \pm c_{j_1}c_{j_k}^*$.

With this notation, we have the following formulae, where the possible dependences between the various $\pm$ signs are not taken into account:

$$\begin{array}{rcl}
c_{i_1} &=& \pm c_{j_1}c_{j_2}^* \\
c_{i_1}^* c_{i_2} &=& (\pm c_{j_1}c_{j_2}^*)^*(\pm c_{j_1}c_{j_3}^*) = \pm c_{j_2}c_{j_3}^* \\
c_{i_2}^* c_{i_3} &=& (\pm c_{j_1}c_{j_3}^*)^*(\pm c_{j_1}c_{j_4}^*) = \pm c_{j_3}c_{j_4}^* \\
\ldots & \ldots & \ldots \\
c_{i_{k-2}}^* c_{i_{k-1}} &=& (\pm c_{j_1}c_{j_{k-1}}^*)^*(\pm c_{j_1}c_{j_k}^*) = \pm c_{j_{k-1}}c_{j_k}^* \\
c_{i_{k-1}}^* &=& (\pm c_{j_1}c_{j_k}^*)^* = \pm c_{j_k}c_{j_1}^*
\end{array}$$

By taking the tensor product of all these formulae, we get:

$$c_{i_1} \otimes c_{i_1}^* c_{i_2} \otimes \ldots \otimes c_{i_{k-2}}^* c_{i_{k-1}} \otimes c_{i_{k-1}}^* = \pm c_{j_1}c_{j_2}^* \otimes c_{j_2}c_{j_3}^* \otimes \ldots \otimes c_{j_k}c_{j_1}^*$$

By applying the linear map given by $a_1 \otimes \ldots \otimes a_k \to a_1 \ldots a_k$ to both sides we see that the sign on the right is actually $+$. That is, we have:

$$c_{i_1} \otimes c_{i_1}^* c_{i_2} \otimes \ldots \otimes c_{i_{k-2}}^* c_{i_{k-1}} \otimes c_{i_{k-1}}^* = c_{j_1}c_{j_2}^* \otimes c_{j_2}c_{j_3}^* \otimes \ldots \otimes c_{j_k}c_{j_1}^*$$

We recognize at right the basic summand in the formula of $R(c_{0_k})$. Now it follows from the definition of i that summing the right terms over all multi-indices $j = (j_1,\ldots,j_k)$ is the same as summing the left terms over all multi-indices $i = (i_1,\ldots,i_{k-1})$, then multiplying by 4. Thus we get:

$$\begin{array}{rcl}
R(c_{0_k}) &=& \dfrac{4}{2}\displaystyle\sum_{i_1\ldots i_{k-1}} c_{i_1} \otimes c_{i_1}^* c_{i_2} \otimes \ldots \otimes c_{i_{k-2}}^* c_{i_{k-1}} \otimes c_{i_{k-1}}^* \\[2ex]
&=& 2\displaystyle\sum_{i_1\ldots i_{k-1}} (c_{i_1} \otimes c_{i_1}^*)_{12}(c_{i_2} \otimes c_{i_2}^*)_{23}\ldots(c_{i_{k-1}} \otimes c_{i_{k-1}}^*)_{k-1,k} \\[2ex]
&=& 2\left(\displaystyle\sum_{i_1} c_{i_1} \otimes c_{i_1}^*\right)_{12}\left(\displaystyle\sum_{i_2} c_{i_2} \otimes c_{i_2}^*\right)_{23}\ldots\left(\displaystyle\sum_{i_{k-1}} c_{i_{k-1}} \otimes c_{i_{k-1}}^*\right)_{k-1,k} \\[2ex]
&=& 2f_{12}f_{23}\ldots f_{k-1,k}
\end{array}$$

Now f being invariant under SU_2, it is the same for each $f_{i,i+1}$. Since the invariants form an algebra, the above product is invariant. This concludes the proof. $\square$

The aim of the next lemma is to prove that the previous lemma holds not only for c_0, but also for c_p.

We first introduce some notations. The Kreweras complement of a partition $p \in NC(k)$ is constructed as follows. Consider the ordered set $\{1, \ldots, k\}$. At right of each index i we put an index i', as to get the following sequence of indices:

$$1, 1', 2, 2', \ldots, k, k'$$

The Kreweras complement p^c is then the largest noncrossing partition of the new index set $\{1', \ldots, k'\}$, such that the union of p and p^c is noncrossing.

Definition 3.1. *For a noncrossing partition p we use the notation*

$$\omega(p) = 2 \prod_{i=1}^{l} \left(\prod_{p=1}^{k_i-1} f_{j_{i,p} j_{i,p+1}} \right)$$

where $\{j_{11} < \ldots < j_{1k_1}\}, \ldots, \{j_{l1} < \ldots < j_{lk_l}\}$ are the blocks of p, with the convention that for $k_i = 1$ the product on the right is by definition 1.

Here we use the element f from Lemma 3.1, and the leg-numbering notation. The element $\omega(p)$ is well-defined, because the products on the right pairwise commute.

In particular in the case of the partition $1_k = \{\{1, \ldots, k\}\}$, we have:

$$\omega(1_k) = 2 \prod_{i=1}^{k-1} f_{i,i+1}$$

Since each f_{ij} is invariant under the adjoint action of SU_2 and since the invariants form an algebra, $\omega(p)$ is also invariant under the adjoint action of SU_2.

Lemma 3.3. *$R(c_p)$ is invariant under the action of SU_2, for any $p \in NC(k)$.*

Proof. The idea is to generalize the proof of the previous lemma. In particular we need to generalize the key formula there, namely:

$$R(c_{0_k}) = 2 f_{12} f_{23} \ldots f_{k-1,k}$$

We claim that for any noncrossing partition p, we have:

$$R(c_p) = \omega(p^c)$$

As a first verification, for the trivial partition $p = 0_k$ we have $p^c = 1_k$, and the formula $R(c_p) = \omega(p^c)$ follows from the above identities. Also, for the rough partition $p = 1_k$ we have $p^c = 0_k$, and the claimed equality follows from:

$$
\begin{aligned}
R(c_{1_k}) &= \sum_i R(c_i \otimes \ldots \otimes c_i) \\
&= \frac{1}{2} \sum_i c_i c_i^* \otimes \ldots \otimes c_i c_i^* \\
&= \frac{1}{2} 4(1 \otimes \ldots \otimes 1) \\
&= \omega(0_k)
\end{aligned}
$$

In the general case, we can use the same method as for Lemma 3.2. We have:

$$
\begin{aligned}
R(c_p) &= R\left(\sum_j \delta_{pj} c_j\right) \\
&= R\left(\sum_{j_1\ldots j_k} \delta_{pj} c_{j_1} \otimes \ldots \otimes c_{j_k}\right) \\
&= \frac{1}{2}\sum_{j_1\ldots j_k} \delta_{pj} c_{j_1} c_{j_2}^* \otimes c_{j_2} c_{j_3}^* \otimes \ldots \otimes c_{j_k} c_{j_1}^*
\end{aligned}
$$

To any multi-index $j = (j_1, \ldots, j_k)$ we associate a multi-index $i = (i_1, \ldots, i_k)$ in the following way: i_1 is such that $c_{i_1} = \pm c_{j_1} c_{j_2}^*$, i_2 is such that $c_{i_2} = \pm c_{j_1} c_{j_3}^*$, and so on up to the last index i_k, which is such that $c_{i_k} = \pm c_{j_k} c_{j_1}^*$.

With this notation, together with the observation that the product of all the above $\pm$ signs is actually $+$ (this can be checked as in the proof of Lemma 3.2), the above formula becomes:

$$
R(c_p) = \frac{1}{2}\sum_j \delta_{pj} c_{i_1} \otimes \ldots \otimes c_{i_k}
$$

Let l be an index in $\{1, \ldots, k\}$ and assume that it is a last element of a block of p^c. Let $l_1 < \ldots < l_x = l$ be the ordered enumeration of the elements of this block. We observe that:

$$
c_{j_{l_1}} \ldots c_{j_{l_x}} = 1
$$

For example, if l and $l+1$ are in the same block of p, then $\{l\}$ is a one-element block in p^c and i_l will be such that $c_{i_l} = \pm c_{j_l} c_{j_{l+1}}^* = c_{j_l} c_{j_l}^* = 1$. The general case works by following the argument of Lemma 3.2. We get:

$$
\sum_i \delta_{pi} c_{j_{l_1}} \otimes \ldots \otimes c_{j_{l_x}} = R(c_{0_x})
$$

The point is that $R(c_p)$ is the product of the above expressions, over the blocks of p^c. If we denote these blocks by $\{l_{11} < \ldots < l_{1x_1}\}, \ldots, \{l_{r1} < \ldots < l_{rx_r}\}$, we get, by using the leg-numbering notation:

$$
\begin{aligned}
R(c_p) &= 2^{1-b}\sum_j \prod_{b=1}^r R(c_{0_{x_b}})_{l_{b1}\ldots l_{bx_b}} \\
&= 2^{1-b}\sum_j \prod_{b=1}^r \omega(1_{x_b})_{l_{b1}\ldots l_{bx_b}} \\
&= \omega(p^c)
\end{aligned}
$$

To illustrate the above proof we perform explicitly the computation in the case of the following partition:

$$
p = \{\{1,5\}, \{2\}, \{3,4\}, \{6\}\}
$$

The Kreweras complement of this partition is $p^c = \{\{1, 2, 4\}, \{3\}, \{5, 6\}\}$, and the above method gives:

$$
\begin{aligned}
R(c_p) &= \frac{1}{2} \sum_{j_1 j_2 j_3 j_4} c_{j_1} c_{j_2^*} \otimes c_{j_2} c_{j_3}^* \otimes 1 \otimes c_{j_3} c_{j_1}^* \otimes c_{j_1} c_{j_4}^* \otimes c_{j_4} c_{j_1}^* \\
&= \frac{1}{2} 4 \sum_{i_1 i_2 i_3} c_{i_1} \otimes c_{i_1}^* c_{i_2} \otimes 1 \otimes c_{i_2}^* \otimes c_{i_3} \otimes c_{i_3}^* \\
&= 2 \sum_{i_1 i_2 i_3} (c_{i_1} \otimes c_{i_1}^* c_{i_2} \otimes 1 \otimes c_{i_2}^* \otimes 1 \otimes 1)(1 \otimes 1 \otimes 1 \otimes 1 \otimes c_{i_3} \otimes c_{i_3}^*) \\
&= 2(f_{12} f_{24}) f_{56} \\
&= \omega(p^c)
\end{aligned}
$$

Now back to the general case, the formula $R(c_p) = \omega(p^c)$ shows that $R(c_p)$ is a product of certain elements f_{ij} obtained from f by acting on the various legs of $M_2(\mathbb{C})^{\otimes k}$, and we can conclude as in the proof of the previous lemma. □

4. Proof of faithfulness

We are now in position of proving the main result. With the technical results from previous section, we can describe the operator $R^* E R$ from Proposition 2.1.

Proposition 4.1. *We have* $R^* E R(c_p) = c_p$.

Proof. The previous lemma gives the following formula:

$$
\begin{aligned}
< R^* E R(c_p), c_j > &= < R^* R(c_p), c_j > \\
&= < R(c_p), R(c_j) > \\
&= \sum_i \delta_{pi} < R(c_i), R(c_j) >
\end{aligned}
$$

On the other hand, we have from definitions:

$$
< c_p, c_j > = \delta_{pj}
$$

Since the elements c_j span the ambient space, what is left to prove is:

$$
\sum_i \delta_{pi} < R(c_i), R(c_j) > = \delta_{pj}
$$

But this can be checked by direct computation. First, from the definition of R, we have the following formula:

$$
\begin{aligned}
< R(c_i), R(c_j) > &= \frac{1}{4} < c_{i_1} c_{i_2}^* \otimes \ldots \otimes c_{i_k} c_{i_1}^*, c_{j_1} c_{j_2}^* \otimes \ldots \otimes c_{j_k} c_{j_1}^* > \\
&= \frac{1}{4} < c_{i_1} c_{i_2}^*, c_{j_1} c_{j_2}^* > \ldots < c_{i_k} c_{i_1}^*, c_{j_k} c_{j_1}^* >
\end{aligned}
$$

In this formula all scalar products are 0, 1 and -1. Now assume that $i_1, \ldots, i_k$ and j_1 are fixed. If the first scalar product is ± 1 then j_2 is uniquely determined, then if the second scalar product is ± 1 then j_3 is uniquely determined as well, and so

on. Thus for all scalar products to be ± 1, the multi-index j is uniquely determined by the multi-index i, up to a possible choice of the first index j_1.

Moreover, each choice of j_1 leads to a multi-index $j = (j_1, \ldots, j_k)$, such that all the scalar products are ± 1. Indeed, once $j_2, \ldots, j_k$ are chosen as to satisfy $c_{i_r} c_{i_{r+1}}^* = \pm c_{j_r} c_{j_{r+1}}^*$ for $r = 1, \ldots, k-1$, by multiplying all these formulae we get $c_{i_1} c_{i_k}^* = \pm c_{j_1} c_{j_k}^*$, which shows that the last scalar product is ± 1 as well.

Summarizing, given a multi-index $i = (i_1, \ldots, i_k)$ and a number $s \in \{1, 2, 3, 4\}$, there is a unique multi-index $j = (j_1, \ldots, j_k)$ with $j_1 = s$, such that all the above scalar products are ± 1. We use the notation $j = i \oplus s$.

In the situation $j = i \oplus s$ we have $< R(c_i), R(c_j) >= \pm 1$, and by applying the linear map given by $a_1 \otimes \ldots \otimes a_k \to a_1 \ldots a_k$ to the formula $c_{i_1} c_{i_2}^* \otimes \ldots \otimes c_{i_k} c_{i_1}^* = \pm c_{j_1} c_{j_2}^* \otimes \ldots \otimes c_{j_k} c_{j_1}^*$ we see that the sign is actually $+$. Thus we have:

$$< R(c_i), R(c_j) >= \begin{cases} 1/4 \text{ if } j = i \oplus s \text{ for some } s \in \{1, 2, 3, 4\} \\ 0 \text{ if not} \end{cases}$$

We can come back now to the missing formula. We have:

$$\sum_i \delta_{pi} < R(c_i), R(c_j) >= \frac{1}{4} \sum_{s=1}^4 \delta_{pi}$$

We claim that we have $\delta_{pi} = \delta_{pj}$, for any partition p. Indeed, this follows from the fact that for $r < s$ we have $i_r = i_s$ if and only if the product of $c_{i_t} c_{i_{t+1}}^*$ over $t = r, \ldots, s-1$ equals ± 1, and a similar statement holds for the multi-index j.

We can now conclude the proof. By using $\delta_{pi} = \delta_{pj}$, we get:

$$\sum_i \delta_{pi} < R(c_i), R(c_j) >= \frac{1}{4} \sum_{s=1}^4 \delta_{pj} = \delta_{pj}$$

But this is the formula that we wanted to prove, so we are done. $\square$

Theorem 4.1. *The Pauli representation of $A_s(4)$ is faithful.*

Proof. We denote as usual by $c_1, \ldots, c_4$ the Pauli matrices, and we let $e_1, \ldots, e_4$ be the standard basis of $\mathbb{C}^4$. For a multi-index $i = (i_1, \ldots, i_k)$, we set:

$$\begin{aligned} e_i &= e_{i_1} \otimes \ldots \otimes e_{i_k} \\ c_i &= c_{i_1} \otimes \ldots \otimes c_{i_k} \end{aligned}$$

Each partition $p \in NC(k)$ creates two tensors, in the following way:

$$\begin{aligned} e_p &= \sum_i \delta_{pi} e_i \\ c_p &= \sum_i \delta_{pi} c_i \end{aligned}$$

Consider the following $4^k \times 4^k$ matrices, with entries labeled by multi-indices i, j:

$$P_{ij} = tr \left(\int \pi_{i_1 j_1} \ldots \pi_{i_k j_k} \right)$$

$$U_{ij} = \left(\int u_{i_1 j_1} \dots u_{i_k j_k} \right)$$

According to Proposition 2.2, we have the following formula:

$$P_{ij} = \; < R^* E R(c_j), c_i >$$

Let Φ be the linear map $(\mathbb{C}^4)^{\otimes k} \to M_2(\mathbb{C})^{\otimes k}$ given by $\Phi(e_i) = c_i$. The above equation can be rephrased as:

$$P_{ij} = \; < \Phi^* R^* E R \Phi(e_j), e_i >$$

According to Proposition 2.1, it is enough to prove that we have $P = U$. But U is the orthogonal projection of $(\mathbb{C}^4)^{\otimes k}$ onto the following space (see [8]):

$$S_e = \operatorname{span}\{e_p \mid p \in NC(k)\}$$

Thus what we have to prove is that $\Phi^* R^* E R \Phi$ is the projection onto S_e. But this is equivalent to proving that $R^* E R$ is the projection onto the following space:

$$S_c = \operatorname{span}\{c_p \mid p \in NC(k)\}$$

We know from Proposition 4.1 that $R^* E R(c_p) = c_p$, for all $p \in NC(k)$. This implies that $R^* E R$ restricted to S_c is the identity. This vector space has dimension the Catalan number C_k, and this is exactly the rank of the operator E. Therefore S_c has to be the image of $R^* E R$. Now since $R^* E R$ is self-adjoint, and is the identity on its image, it is the orthogonal projection onto S_c, and this concludes the proof. $\square$

5. DIAGONAL COEFFICIENTS

We can use the Pauli representation for computing laws of certain diagonal coefficients of u, the fundamental corepresentation of $A_s(4)$. These diagonal coefficients are introduced in [7], [8]. We are particularly interested in the following elements:

Definition 5.1. *Associated to $s = 1, 2, 3, 4$ is the average*

$$M_s = \frac{1}{s} \left(u_{11} + u_{22} + \dots + u_{ss} \right)$$

where u is the fundamental corepresentation of $A_s(4)$.

We use notions from free probability from [15], [16]. Recall first that the free Poisson law is the following probability measure on $[0, 4]$:

$$\nu = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} \, dx$$

In this paper we use probability measures supported on $[0, 1]$. So, consider the corresponding compression of the free Poisson law:

$$\nu_1 = \frac{2}{\pi} \sqrt{x^{-1} - 1} \, dx$$

We denote by λ_1 the Lebesgue measure on $[0, 1]$.

Lemma 5.1. *We have the following formulae, where $(a, b, c, d) \in S^3$:*
  (1) $\operatorname{law}(a^2) = \nu_1$.

(2) $\mathrm{law}(a^2 + b^2) = \lambda_1$.

(3) $\mathrm{law}(a^2 + b^2 + c^2 + d^2) = \delta_1$.

Proof. These results are all well-known:

(1) The variable $2a$ is semicircular on $[-2, 2]$, as one can see geometrically on S^3, or by using representation theory of SU_2, so its square $4a^2$ is free Poisson.

(2) This follows from the fact that when projecting S^3 on the unit disk, the uniform measure on S^3 becomes the uniform measure on the unit disk.

(3) This follows from $a^2 + b^2 + c^2 + d^2 = 1$. $\square$

With these formulae in hand, we can compute the laws of M_1, M_2, M_4. Modulo a Dirac mass at 0, these are the Dirac, Lebesgue and free Poisson laws:

Theorem 5.1. *For $s = 1, 2, 4$ we have the formula*

$$\mathrm{law}(M_s) = \left(1 - \frac{s}{4}\right)\delta_0 + \frac{s}{4}\mu_s$$

where $\mu_1 = \delta_1$, $\mu_2 = \lambda_1$, $\mu_4 = \nu_1$.

Proof. The $s = 1$ assertion is clear, and the $s = 4$ one is known from [2]. For reasons of uniformity of the proof, we will prove these assertions as well.

Consider an element $x \in SU_2$, and write it as $x = x_1 c_1 + \ldots + x_4 c_4$. With the notations $c_i^2 = \varepsilon_i$ and $c_i c_j = \varepsilon_{ij} c_j c_i$, with $\varepsilon_i, \varepsilon_{ij} \in \{\pm 1\}$, we have:

$$c_i x c_i = \sum_j \varepsilon_i \varepsilon_{ij} x_j c_j$$

This gives the following formula for the orthogonal projection onto the vector $c_i x c_i$, denoted π_{ii}, as in previous section:

$$< \pi_{ii} c_j, c_k > = \varepsilon_{ij} \varepsilon_{ik} x_j x_k$$

Now by using the more convenient notation $x = ac_1 + bc_2 + cc_3 + dc_4$, we get from the multiplication table of Pauli matrices the following formulae:

$$\pi_{11} = \begin{pmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{pmatrix} \qquad \pi_{22} = \begin{pmatrix} a^2 & ab & -ac & -ad \\ ab & b^2 & -bc & -bd \\ -ac & -bc & c^2 & cd \\ -ad & -bd & cd & d^2 \end{pmatrix}$$

$$\pi_{33} = \begin{pmatrix} a^2 & -ab & ac & -ad \\ -ab & b^2 & -bc & bd \\ ac & -bc & c^2 & -cd \\ -ad & bd & -cd & d^2 \end{pmatrix} \qquad \pi_{44} = \begin{pmatrix} a^2 & -ab & -ac & ad \\ -ab & b^2 & bc & -bd \\ -ac & bc & c^2 & -cd \\ ad & -bd & -cd & d^2 \end{pmatrix}$$

We have $M_1 = \pi_{11}$, and by making averages we get M_2, M_4:

$$M_2 = \begin{pmatrix} a^2 & ab & 0 & 0 \\ ab & b^2 & 0 & 0 \\ 0 & 0 & c^2 & cd \\ 0 & 0 & cd & d^2 \end{pmatrix} \qquad M_4 = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & d^2 \end{pmatrix}$$

With these notations, we have to compute the following numbers:

$$\int \text{tr}(M_s^k) = \int \left(\frac{u_{11} + \ldots + u_{ss}}{s} \right)^k$$

We first compute the characteristic polynomial of each M_s:

$$
\begin{aligned}
\det(y - M_1) &= y^3(y - 1) \\
\det(y - M_2) &= y^2(y - a^2 - b^2)(y - c^2 - d^2) \\
\det(y - M_4) &= (y - a^2)(y - b^2)(y - c^2)(y - d^2)
\end{aligned}
$$

Thus we have the following diagonalisations:

$$
\begin{aligned}
M_1 &\sim \text{diag}(0, 0, 0, 1) \\
M_2 &\sim \text{diag}(0, 0, a^2 + b^2, c^2 + d^2) \\
M_4 &\sim \text{diag}(a^2, b^2, c^2, d^2)
\end{aligned}
$$

We take powers, we apply the trace, and we integrate:

$$
\begin{aligned}
\int \text{tr}(M_1^k) &= \frac{1}{4} \\
\int \text{tr}(M_2^k) &= \frac{1}{4} \int (a^2 + b^2)^k + (c^2 + d^2)^k \\
\int \text{tr}(M_4^k) &= \frac{1}{4} \int a^{2k} + b^{2k} + c^{2k} + d^{2k}
\end{aligned}
$$

By symmetry reasons we have:

$$
\begin{aligned}
\int \text{tr}(M_1^k) &= \frac{1}{4} \\
\int \text{tr}(M_2^k) &= \frac{1}{2} \int (a^2 + b^2)^k \\
\int \text{tr}(M_4^k) &= \int a^{2k}
\end{aligned}
$$

The result follows now from Lemma 5.1. $\qquad \square$

We would like to end this section with a technical result, to be used later on.

Proposition 5.1. *The Cauchy transforms for M_1, M_2, M_4 are:*

$$
\begin{aligned}
G_1(\xi) &= \frac{1}{\xi} + \frac{1}{4(\xi^2 - \xi)} \\
G_2(\xi) &= \frac{1}{2} \left(\frac{1}{\xi} - \log \left(1 - \frac{1}{\xi} \right) \right) \\
G_4(\xi) &= 2 \left(1 - \sqrt{1 - \frac{1}{\xi}} \right)
\end{aligned}
$$

Proof. This follows from the above formulae. $\qquad \square$

6. Numeric results

In previous section we computed the law of the average M_s, with $s = 1, 2, 4$. The same proof doesn't apply to the missing variable M_3, because the corresponding matrix cannot be diagonalized explicitly. This technical problem is to be related to the general principle "the sphere cannot be cut in three parts".

The undiagonalizable (or cutting) matrix is as follows:

Proposition 6.1. M_3 *has the same law as the random matrix*

$$M_3 = \frac{1}{3} \begin{pmatrix} 3a^2 & -ab & -ac & -ad \\ -ab & 3b^2 & -bc & -bd \\ -ac & -bc & 3c^2 & -cd \\ -ad & -bd & -cd & 3d^2 \end{pmatrix}$$

depending on $x = (a, b, c, d)$ *on the sphere* S^3.

Proof. By symmetry reasons M_3 has the same law as $(u_{22} + u_{33} + u_{44})/3$, which has in turn the same law as the following matrix:

$$M_3 = \frac{1}{3} \left(\pi_{22} + \pi_{33} + \pi_{44} \right)$$

By using the formulae of π_{ii}, we get the matrix in the statement. $\square$

Observe that M_3 has trace 1, and has $(a^{-1}, b^{-1}, c^{-1}, d^{-1})$ as 0-eigenvector. Thus the characteristic polynomial of M_3 is of the following form:

$$Q(y) = y^4 - y^3 + Ky^2 - Ly$$

An explicit computation gives the following formulae for K, L:

$$K = \frac{8}{9} \left(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 \right)$$

$$L = \frac{16}{27} \left(a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 \right)$$

In principle, this can be used for computing the Cauchy transform:

$$G(\xi) = \frac{1}{4} \int \frac{Q'(\xi)}{Q(\xi)} \, dx$$

The corresponding integration problem on S^3 looks particularly difficult, and we don't know how to solve it. However, we did a lot of related abstract or numeric computations, and we have the following result:

Theorem 6.1. *The first moments of* $N_3 = 3M_3$ *are given by:*

order	1	2	3	4	5	6	...
moment	3/4	5/4	5/2	109/20	25/2	4157/140	...

...	7	8	9
...	1449/20	75877/420	64223/140

Proof. We have to compute the moments of the matrix $N_3(a, b, c, d)$, where (a, b, c, d) is uniformly distributed along the sphere S^3.

For this, let (a', b', c', d') be independent standard Gaussian variables. We have the following equality of joint distributions:

$$\text{law}(a', b', c', d') = \text{law}(\rho a, \rho b, \rho c, \rho d)$$

where ρ is the positive square root of a standard chi-square distribution of parameter 4, independent from (a, b, c, d).

Consider now the random matrix $N_3(a', b', c', d')$, obtained by replacing (a, b, c, d) with (a', b', c', d'). Since ρ is a real variable independent from (a, b, c, d), whose even moments are the numbers $2^k(k+1)!$, we have the following computation:

$$
\begin{aligned}
\int N_3(a', b', c', d')^k &= \int N_3(\rho a, \rho b, \rho c, \rho d)^k \\
&= \int \rho^{2k} N_3(a, b, c, d)^k \\
&= \int \rho^{2k} \int N_3(a, b, c, d)^k \\
&= 2^k(k+1)! \int N_3(a, b, c, d)^k
\end{aligned}
$$

The integrals on the left can be computed with Maple, and we get the result. $\square$

The above theorem shows that our matrix model has also a computational interest: indeed, a direct attempt to make the above computations only with the Weingarten function summation formulae (or even with the matrix model but without the above Gaussianization trick) could not yield more than 5-6 moments whereas the method described in the proof of the above theorem could easily yield up to 15 moments.

Yet, these computations don't shed any light on where these moments come from: for instance the number 64223, appearing in the above table, is prime.

As a last comment, we have the following negative result.

Proposition 6.2. $\text{law}(M_3) \neq \frac{1}{4}\delta_0 + \frac{3}{4}\text{law}\left(a^2 + b^2 + c^2\right)$.

Indeed, the second moments of these laws are respectively 5/36 and 15/32. This can be checked by a routine computation, and contradicts what one might want to conjecture, after a quick comparison of Lemma 5.1 and Theorem 5.1.

7. Lebesgue-Dirac interpolation

In this section and in the next one we perform some technical computations. Our motivation is as follows. Consider the variables $u_{11} + \ldots + u_{ss}$ with $s = 1, 2, 4$, whose laws are known. The variable $u_{11} + u_{22} + u_{33}$ has the same law as:

$$u_{22} + u_{33} + u_{44} = (u_{11} + u_{22} + u_{33} + u_{44}) - u_{11}$$

Thus, we are in front of the following problem: we know how to compute the laws of 3 variables, and we want to compute the law of a certain 4-th variable, belonging

the same projective plane. This can be regarded as being part of the more general problem of finding the law of an arbitrary point in the plane.

We will work out here two simple computations in this sense.

We know how to compute the laws of the following two elements:

$$
\begin{aligned}
w_0 &= (u_{11} + u_{22})/2 \\
w_1 &= u_{11}
\end{aligned}
$$

We can consider the following element, interpolating between them:

Definition 7.1. *To any real number t we associate the element*

$$
w_t = \frac{1+t}{2} \cdot u_{11} + \frac{1-t}{2} \cdot u_{22}
$$

where u is the fundamental corepresentation of $A_s(4)$.

By using the matrix formulae in section 5, the characteristic polynomial of w_t is given by $P(y) = y^2 Q(y)$, where Q is the following degree two polynomial:

$$
Q(y) = y^2 - y + (1 - t^2)(a^2 + b^2)(c^2 + d^2)
$$

In other words, w_t has a double 0 eigenvalue, and a 2×2 matrix block. The law of the matrix block can be computed by using the following lemma.

Lemma 7.1. *The Cauchy transform of a 2×2 random matrix M having characteristic polynomial $Q(y) = y^2 - By + C$ is given by the following formula:*

$$
G(\xi) = \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-1)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int B^q C^p
$$

Proof. The eigenvalues of M are the roots of Q, so we get:

$$
\begin{aligned}
\int \mathrm{tr}(M^k) &= 2^{-k-1} \int (B + \sqrt{B^2 - 4C})^k + (B - \sqrt{B^2 - 4C})^k \\
&= 2^{-k} \sum_{n=0}^{k/2} \binom{k}{2n} \int B^{k-2n}(B^2 - 4C)^n \\
&= 2^{-k} \sum_{n=0}^{k/2} \sum_{p=0}^{n} (-4)^p \binom{k}{2n} \binom{n}{p} \int B^{k-2p} C^p
\end{aligned}
$$

Here a sum from 0 to a real number r means by definition sum from 0 to the integral part of r. We compute now the Cauchy transform:

$$
\begin{aligned}
G(\xi) &= \sum_{k=0}^{\infty} \xi^{-k-1} \int \mathrm{tr}(M^k) \\
&= \frac{1}{\xi} \sum_{k=0}^{\infty} \sum_{n=0}^{k/2} \sum_{p=0}^{n} \frac{(-4)^p}{(2\xi)^k} \binom{k}{2n} \binom{n}{p} \int B^{k-2p} C^p
\end{aligned}
$$

We make the replacements $n = p + m$ and $k = 2p + q$:

$$G(\xi) = \frac{1}{\xi} \sum_{p,q=0}^{\infty} \sum_{m=0}^{q/2} \frac{(-4)^p}{(2\xi)^{2p+q}} \binom{2p+q}{2p+2m} \binom{p+m}{p} \int B^q C^p$$

We use now the following standard identity, valid for $p + q > 0$:

$$\sum_{m=0}^{q/2} \binom{2p+q}{2p+2m} \binom{p+m}{p} = 2^{q-1} \frac{2p+q}{p+q} \binom{p+q}{q}$$

This gives the following formula:

$$G(\xi) = \frac{1}{\xi} + \frac{1}{\xi} \sum_{p+q>0} \frac{(-4)^p \cdot 2^{q-1}}{(2\xi)^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int B^q C^p$$

$$= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-1)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int B^q C^p$$

This gives the formula in the statement. □

Theorem 7.1. *The Cauchy transform of the law of w_t is given by*

$$G(\xi) = \frac{1}{2\xi} + \frac{1-2\xi}{\xi-\xi^2} \cdot \frac{\text{arcsinh}(\sqrt{x}/2)}{\sqrt{4x+x^2}}$$

where $x = (1-t^2)/(\xi^2 - \xi)$.

Proof. We recall that the characteristic polynomial of w_t is given by the formula $P(y) = y^2 Q(y)$, where Q is the following degree two polynomial:

$$Q(y) = y^2 - y + (1-t^2)(a^2+b^2)(c^2+d^2)$$

Thus the Cauchy transform of w_t is the average between $1/\xi$ and the Cauchy transform G_B of the corresponding 2×2 matrix block:

$$G(\xi) = \frac{1}{2} \left(\frac{1}{\xi} + G_B(\xi) \right)$$

We apply Lemma 7.1 with the above characteristic polynomial, which is of the form $Q(y) = y^2 - y + sD$, with $s = 1 - t^2$ and $D = (a^2+b^2)(c^2+d^2)$. We get:

$$G_B(\xi) = \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int D^p$$

Thus we get the following formula:

$$G(\xi) = \frac{1}{\xi} + \frac{1}{4\xi} \sum_{p+q>0} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int D^p$$

We can get rid of $p + q > 0$ by using the Cauchy transform G_0:

$$G(\xi) = G_0(\xi) + \frac{1}{4\xi} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int D^p$$

$$= \; G_0(\xi) + \frac{1}{4\xi} \sum_{p=1}^{\infty} \left(-\frac{s}{\xi} \right)^p \sum_{q=0}^{\infty} \frac{1}{\xi^{p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int D^p$$

$$= \; G_0(\xi) + \frac{1}{4\xi} \sum_{p=1}^{\infty} \left(-\frac{s}{\xi} \right)^p \frac{2\xi - 1}{(\xi - 1)^{p+1}} \int D^p$$

$$= \; G_0(\xi) + \frac{1 - 2\xi}{4(\xi - \xi^2)} \sum_{p=1}^{\infty} \left(\frac{s}{\xi - \xi^2} \right)^p \int D^p$$

We use the following formula for G_0, coming from Proposition 5.1:

$$G_0(\xi) = \frac{1}{2\xi} + \frac{1 - 2\xi}{4(\xi - \xi^2)}$$

Also, from $a^2 + b^2 + c^2 + d^2 = 1$ we get $D = T - T^2$ with $T = a^2 + b^2$, and we know from Lemma 5.1 that $\text{law}(T)$ is the Lebesgue measure on $[0, 1]$, so:

$$\sum_{p=1}^{\infty} (-x)^p \int D^p \; = \; \sum_{p=1}^{\infty} (-x)^p \sum_{k=0}^{p} \binom{p}{k} \int T^p (-T)^k$$

$$= \; \sum_{p=1}^{\infty} (-x)^p \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{p+k+1}$$

$$= \; -1 + \frac{4 \operatorname{arcsinh}(\sqrt{x}/2)}{\sqrt{4x + x^2}}$$

With $x = s/(\xi^2 - \xi)$, we get the formula in the statement. $\qquad\square$

We should mention that at $t = 0, 1$ the formula in Theorem 7.1 gives indeed those in Proposition 5.1. At $t = 0$ this follows from $\operatorname{arcsinh}(y) = \log(y + \sqrt{1 + y^2})$, and at $t = 1$ this follows from $\operatorname{arcsinh}(x) \sim x$ for $x \sim 0$.

The other remark is that Theorem 7.1 can be combined with the Stieltjes inverse formula, in order to get the density of the law of w_t. The corresponding function is piecewise analytic, and the precise formulae will not be given here.

8. Poisson-Lebesgue interpolation

We perform here a second computation, which is slightly more technical than the one in the previous section. Consider the following two elements:

$$v_0 \; = \; \frac{1}{4}(u_{11} + u_{22} + u_{33} + u_{44})$$
$$v_1 \; = \; \frac{1}{2}(u_{11} + u_{22})$$

We can consider the following element, interpolating between them:

Definition 8.1. *To any real number t we associate the element*

$$v_t = \frac{1+t}{4}(u_{11} + u_{22}) + \frac{1-t}{4}(u_{33} + u_{44})$$

where u is the fundamental corepresentation of $A_s(4)$.

The characteristic polynomial of v_t can be computed by using the matrix formulae in section 5. We get that this is the product of the following polynomials:

$$Q_1(y) = y^2 - (a^2 + b^2)y + (1 - t^2)a^2b^2$$
$$Q_2(y) = y^2 - (c^2 + d^2)y + (1 - t^2)c^2d^2$$

Thus v_t decomposes as a sum of two 2×2 matrix blocks, and we can compute its law, provided that we know how to integrate polynomials in a, b.

We denote by $x = (a, b, c, d)$ the points on the real sphere S^3.

Lemma 8.1. *We have the formula*

$$\int a^{2k-2p}b^{2p} = \frac{4^{-k}}{(k+1)!} \cdot \frac{(2p)!(2k-2p)!}{p!(k-p)!}$$

where the integral is with respect to the uniform measure on S^3.

Proof. We denote by c_{rp} the numbers in the statement, where $r = k - p$:

$$c_{rp} = \int a^{2r}b^{2p}$$

Consider the following variables, depending on a real parameter t:

$$A = \cos t \cdot a + \sin t \cdot b$$
$$B = -\sin t \cdot a + \cos t \cdot b$$

Since the map $(a, b) \to (A, B)$ is a rotation, we have:

$$c_{rp} = \int A^{2r}B^{2p}$$

We use the following formula, coming from $A' = B$ and $B' = -A$:

$$(A^{2r+1}B^{2p-1})' = (2r+1)A^{2r}B^{2p} - (2p-1)A^{2r+2}B^{2p-2}$$

Now since each c_p doesn't depend on t, the integral of $A^{2r+1}B^{2p-1}$ doesn't depend on t either. Thus the derivative of this integral vanishes, and we get:

$$(2r+1)\int A^{2r}B^{2p} = (2p-1)\int A^{2r+2}B^{2p-2}$$

This gives the following formula for the numbers c_{rp}:

$$\begin{aligned} c_{rp} &= \frac{2p-1}{2r+1}c_{r+1,p-1} \\ &= \frac{(2p-1)(2p-3)\ldots(2p-2s+1)}{(2r+1)(2r+3)\ldots(2r+2s-1)}c_{r+s,p-s} \\ &= \frac{(2p-1)(2p-3)\ldots 1}{(2r+1)(2r+3)\ldots(2k-1)}c_{k0} \end{aligned}$$

The number c_{k0} is the k-th moment of a, so it is 4^{-k} times the k-th Catalan number, because $2a$ is known to be semicircular. This gives the following formula:

$$c_{rp} = \frac{(2p)!}{2^p\,p!}\left(\frac{2^k\,k!}{(2k)!} \cdot \frac{(2r)!}{2^r\,r!}\right)\left(4^{-k} \cdot \frac{(2k)!}{k!(k+1)!}\right)$$

$$= \frac{4^{-k}}{(k+1)!} \cdot \frac{2^k (2p)!(2r)!}{2^{p+r} p! r!}$$

With $r = k - p$, this gives the formula in the statement. $\qquad\square$

Theorem 8.1. *The Cauchy transform of the law of v_t is given by:*

$$G'(\xi) = \frac{1}{2} \cdot \frac{2\xi - 1}{\xi^2 - \xi^3} \sqrt{\frac{\xi - \xi^2}{\xi - \xi^2 - (1 - t^2)/4}}$$

Proof. By symmetry reasons the two blocks of v_t have the same spectral measure, so the Cauchy transform of v_t is equal to the Cauchy transform of each block, say of the first block. We can apply Lemma 7.1, and we get:

$$
\begin{aligned}
G(\xi) &= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-1)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \int (a^2 + b^2)^q (sa^2 b^2)^p \\
&= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \sum_{r=0}^{q} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{q} \binom{q}{r} \int a^{2p+2q-2r} b^{2p+2r} \\
&= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \sum_{r=0}^{q} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} K
\end{aligned}
$$

Here $s = 1 - t^2$, and K is the following product, obtained by expanding all binomial coefficients, then by multiplying by the quantity in Lemma 8.1:

$$
\begin{aligned}
K &= \frac{(p+q)!}{p! q!} \cdot \frac{q!}{r!(q-r)!} \cdot \frac{4^{-2p-q}}{(2p+q+1)!} \cdot \frac{(2p+2r)!(2p+2q-2r)!}{(p+r)!(p+q-r)!} \\
&= \frac{4^{-2p-q}(p+q)!}{p!(2p+q+1)!} \cdot \frac{(2p+2r)!(2p+2q-2r)!}{r!(q-r)!(p+r)!(p+q-r)!}
\end{aligned}
$$

We use now the following standard identity:

$$\sum_{r=0}^{q} \frac{(2p+2r)!(2p+2q-2r)!}{r!(q-r)!(p+r)!(p+q-r)!} = \frac{4^q}{q!} \cdot \frac{(2p)!(2p+q)!}{p!p!}$$

Thus when summing K over r, we get the following quantity:

$$
\begin{aligned}
K_r &= \frac{4^{-2p-q}(p+q)!}{p!(2p+q+1)!} \cdot \frac{4^q}{q!} \cdot \frac{(2p)!(2p+q)!}{p!p!} \\
&= \frac{16^{-p}}{2p+q+1} \cdot \frac{(2p)!(p+q)!}{q!p!p!p!}
\end{aligned}
$$

We can get back now to the Cauchy transform:

$$
\begin{aligned}
G(\xi) &= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} K_r \\
&= \frac{1}{\xi} + \frac{1}{2\xi} \sum_{p+q>0} \frac{(-s)^p}{\xi^{2p+q}} \cdot \frac{2p+q}{p+q} \cdot \frac{16^{-p}}{2p+q+1} \cdot \frac{(2p)!(p+q)!}{q!p!p!p!}
\end{aligned}
$$

$$= \quad \frac{1}{\xi} + \frac{1}{2} \sum_{p+q>0} \left(-\frac{s}{16}\right)^p \frac{1}{\xi^{2p+q+1}} \cdot \frac{2p+q}{2p+q+1} \cdot \frac{(2p)!(p+q-1)!}{q!p!p!p!}$$

We can get rid of $p + q > 0$ by using G_0, the value of G at $s = 0$:

$$G(\xi) \quad = \quad G_0(\xi) + \frac{1}{2} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \left(-\frac{s}{16}\right)^p \frac{1}{\xi^{2p+q+1}} \cdot \frac{2p+q}{2p+q+1} \cdot \frac{(2p)!(p+q-1)!}{q!p!p!p!}$$

$$= \quad G_0(\xi) + \frac{1}{2} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \left(-\frac{s}{16}\right)^p \frac{(2p+q+1)^{-1}}{\xi^{2p+q+1}} \cdot \frac{2p+q}{p+q} \binom{2p}{p} \binom{p+q}{p}$$

We take the derivative with respect to ξ:

$$G'(\xi) \quad = \quad G_0'(\xi) - \frac{1}{2} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \left(-\frac{s}{16}\right)^p \frac{1}{\xi^{2p+q+2}} \cdot \frac{2p+q}{p+q} \binom{2p}{p} \binom{p+q}{p}$$

$$= \quad G_0'(\xi) - \frac{1}{2} \sum_{p=1}^{\infty} \left(-\frac{s}{16}\right)^p \frac{1}{\xi^{p+2}} \binom{2p}{p} \sum_{q=0}^{\infty} \frac{1}{\xi^{p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{p}$$

We use now the following standard identity:

$$\sum_{q=0}^{\infty} \frac{1}{\xi^{p+q}} \cdot \frac{2p+q}{p+q} \binom{p+q}{p} = \frac{2\xi-1}{(\xi-1)^{p+1}}$$

This gives the following formula for the Cauchy transform:

$$G'(\xi) \quad = \quad G_0'(\xi) - \frac{1}{2} \sum_{p=1}^{\infty} \left(-\frac{s}{16}\right)^p \frac{1}{\xi^{p+2}} \binom{2p}{p} \frac{2\xi-1}{(\xi-1)^{p+1}}$$

$$= \quad G_0'(\xi) + \frac{1}{2} \sum_{p=1}^{\infty} \left(\frac{s}{16}\right)^p \frac{1}{\xi^{p+2}} \binom{2p}{p} \frac{2\xi-1}{(1-\xi)^{p+1}}$$

$$= \quad G_0'(\xi) + \frac{2\xi-1}{2\xi^2-2\xi^3} \sum_{p=1}^{\infty} \left(\frac{s/16}{\xi-\xi^2}\right)^p \binom{2p}{p}$$

We use now the following standard identity:

$$\sum_{p=0}^{\infty} x^p \binom{2p}{p} = \frac{1}{\sqrt{1-4x}}$$

This gives the following formula for the Cauchy transform:

$$G'(\xi) \quad = \quad G_0'(\xi) + \frac{2\xi-1}{2\xi^2-2\xi^3} \left(\left(1 - 4 \cdot \frac{s/16}{\xi-\xi^2}\right)^{-1/2} - 1 \right)$$

$$= \quad G_0'(\xi) + \frac{2\xi-1}{2\xi^2-2\xi^3} \left(\frac{\xi-\xi^2}{\xi-\xi^2-s/4}\right)^{1/2} - \frac{2\xi-1}{2\xi^2-2\xi^3}$$

We compute G'_0 by using Proposition 5.1, and we get:

$$G'_0(\xi) = -\frac{1}{2} \cdot \frac{2\xi - 1}{\xi^3 - \xi^2}$$

This gives the following formula for the Cauchy transform:

$$G'(\xi) = \frac{1}{2} \cdot \frac{2\xi - 1}{\xi^2 - \xi^3} \sqrt{\frac{\xi - \xi^2}{\xi - \xi^2 - s/4}}$$

Together with $s = 1 - t^2$, we get the formula in the statement.   □

9. SYMMETRIC GROUPS

Let u be the fundamental corepresentation of $C(S_4)$. We consider the following element of $C(S_4)$, depending on real parameters t_i which sum up to 1:

$$u_t = t_1 u_{11} + t_2 u_{22} + t_3 u_{33} + t_4 u_{44}$$

These can be regarded as being "classical analogues" of the variables considered in the previous sections. Their laws can be computed as follows.

Theorem 9.1. *The law of u_t is the following average of Dirac masses:*

$$\mathrm{law}(u_t) = \frac{1}{24} \left(9\delta_0 + \delta_1 + 2 \sum_i \delta_{t_i} + \sum_{i \neq j} \delta_{t_i + t_j} \right)$$

Proof. We have $u_{ii} = \chi(X_i)$, where X_i is the set of permutations in S_4 fixing i. There are 6 such permutations, namely:

(1) The identity I.
(2) The two 3-cycles fixing i; we denote by C_i the set they form.
(3) The three transpositions fixing i; we denote by T_i the set they form.

Observe that: the identity I belongs to each X_i; the set C_i doesn't intersect the set X_j, for $j \neq i$; each of the six transpositions in S_4 can be obtained by taking intersections between the sets T_i and their complements.

These remarks show that the algebra Δ generated by the diagonal elements u_{ii} is a 12-dimensional vector space, with the following basis:

(1) The projection $\chi\{I\}$.
(2) The 4 projections $\chi\{C_i\}$.
(3) The 6 projections $\chi\{T_{ij}\}$, where T_{ij} is the transposition fixing $i \neq j$.
(4) The projection $\chi\{D\}$ onto what's left.

With these notations, we have the following formula:

$$u_{ii} = \chi\{I\} + \chi\{C_i\} + \sum_{i \neq j} \chi\{T_{ij}\}$$

We get in this way a formula for average in the statement:

$$u_t = \chi\{I\} + \sum_i t_i \chi\{C_i\} + \sum_{i \neq j} (t_i + t_j) \chi\{T_{ij}\}$$

On the other hand, the restriction of the integration (or averaging) over S_4 to the subalgebra $\Delta \subset \mathbb{C}(S_4)$ is obtained by counting elements in various subsets of S_4 corresponding to the above basis of Δ. We get:

$$\int \chi\{I\} = \frac{1}{24}, \int \chi\{C_i\} = \frac{1}{12}, \int \chi(T_{ij}) = \frac{1}{24}, \int \chi\{D\} = \frac{9}{24}$$

We can compute now the moments of u_t:

$$\begin{aligned}
\int u_t^k &= \int \chi\{I\} + \sum_i t_i^k \, \chi\{C_i\} + \sum_{i \neq j} (t_i + t_j)^k \, \chi\{T_{ij}\} \\
&= \frac{1}{24} + \frac{1}{12} \sum_i t_i^k + \frac{1}{24} \sum_{i \neq j} (t_i + t_j)^k \\
&= \frac{1}{24}\left(1^k + 2 \sum_i t_i^k + \sum_{i \neq j} (t_i + t_j)^k + 9 \cdot 0^k \right)
\end{aligned}$$

In this formula the $9 \cdot 0^k = 0$ term is there for the 9 coefficient to produce the equality $24 = 1 + 2 \cdot 4 + 6 + 9$. This gives the formula in the statement. $\qquad \square$

Corollary 9.1. *The laws of the averages $m_s = (u_{11} + \ldots + u_{ss})/s$ are:*

$$\begin{aligned}
\mathrm{law}(m_1) &= \frac{1}{24}\left(18\delta_0 + 6\delta_1 \right) \\
\mathrm{law}(m_2) &= \frac{1}{24}\left(14\delta_0 + 8\delta_{1/2} + 2\delta_1 \right) \\
\mathrm{law}(m_3) &= \frac{1}{24}\left(11\delta_0 + 9\delta_{1/3} + 3\delta_{2/3} + \delta_1 \right) \\
\mathrm{law}(m_4) &= \frac{1}{24}\left(9\delta_0 + 8\delta_{1/4} + 6\delta_{1/2} + \delta_1 \right)
\end{aligned}$$

The challenging problem here is to work out the analogy with $A_s(4)$. It is known from [8] that the analogy between classical and quantum appears in the limit $n \to \infty$, with the Poisson semigroup of measures for $C(S_n)$ corresponding to the free Poisson semigroup for $A_s(n)$. The above computations should be regarded as a first step towards understanding what happens with the analogy, when n is fixed.

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